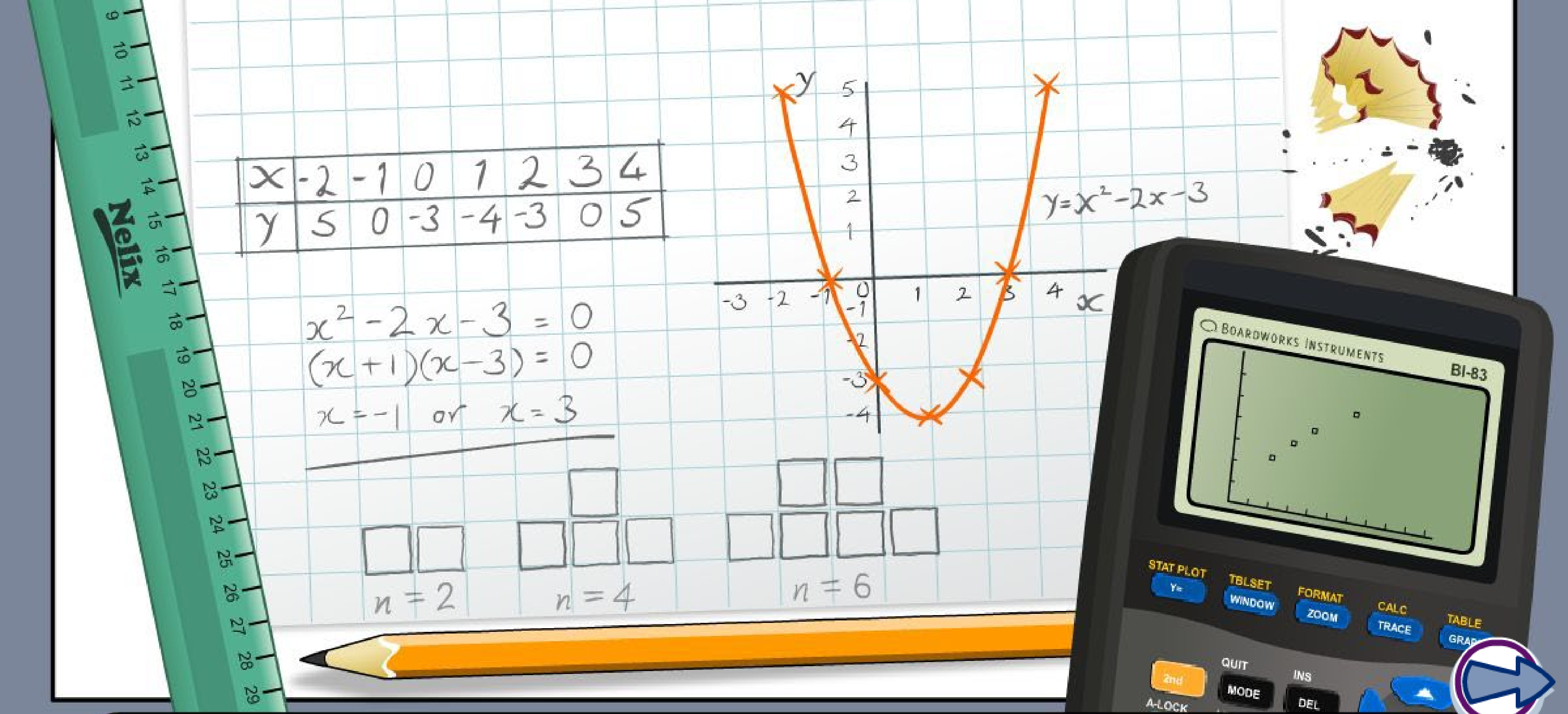
Completing the square



Information

Common core icons

This icon indicates a slide where the Standards for Mathematical Practice are being developed. Details of these are given in the Notes field.

Slides containing examples of mathematical modeling are marked with this stamp.

This icon indicates an opportunity for discussion or group work.

The **Standards for Mathematical Practice** outlined in the Common Core State Standards for Mathematics describe varieties of expertise that mathematics educators at all levels should seek to develop in their students.

They are:

1) Make sense of problems and persevere in solving them.

2) Reason abstractly and quantitatively.

3) Construct viable arguments and critique the reasoning of others.

4) Model with mathematics.

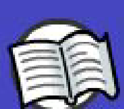
5) Use appropriate tools strategically.

6) Attend to precision.

7) Look for and make use of structure.

8) Look for and express regularity in repeated reasoning.

This icon indicates that the slide contains activities created in Flash. These activities are not editable.



This icon indicates teacher's notes in the Notes field.

Perfect squares

Some quadratic expressions can be written as perfect squares. For example:

$x^2 + 2x + 1 = (x + 1)^2$

$x^2 + 4x + 4 = (x + 2)^2$

$x^2 + 6x + 9 = (x + 3)^2$

$x^2 - 2x + 1 = (x - 1)^2$

$x^2 - 4x + 4 = (x - 2)^2$

$x^2 - 6x + 9 = (x - 3)^2$

In general,

$x^2 + 2ax + a^2 = (x + a)^2$ and $x^2 - 2ax + a^2 = (x - a)^2$

How could the quadratic expression $x^2 + 6x$ be made into a perfect square?

We could add 9 to it.

Adding 9 to the expression $x^2 + 6x$ to make it into a perfect square is called **completing the square**.

We can write: $x^2 + 6x = x^2 + 6x + 9 - 9$

We added 9 so we have to subtract it again to keep both sides equal.

Since $x^2 + 6x + 9$ can be written as perfect square, we write:

$$x^2 + 6x = (x + 3)^2 - 9$$

Completing the square:

$$x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$$

Investigate completing the square

We 'complete the square' for a quadratic expression to make it into a perfect square, which can be factored easily.

The general formula for completing the square is:

$$x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$$

But why?

Press **play** to see a graphical explanation of completing the square method.

Completing the square

In general,

$$x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$$

Complete the square for $x^2 + 10x$.

use the equation: $(x + 5)^2 - 5^2$

simplify: $\mathbf{(x + 5)^2 - 25}$

Compare this expression to
$(x + 5)^2 = x^2 + 10x + 25$

Complete the square for $x^2 - 3x$.

use the equation: $(x - \frac{3}{2})^2 - (-\frac{3}{2}{}^2)$

simplify: $\mathbf{(x - \frac{3}{2})^2 - \frac{9}{4}}$

Compare this expression to
$(x - 1.5)^2 = x^2 - 3x + 2.25$

Adjust the variables using the blue arrows.

Complete the square for $x^2 + 5x$.

$$x^2 + 5x = (x + 0)^2 - 0$$

How can we complete the square for $x^2 + 8x + 9$?

Notice that this quadratic has a constant value added to it.

We can complete the square as normal, we just have to remember to add on the constant value.

Compare this to
$(x + 4)^2 = x^2 + 8x + 16$

complete the square: $(x^2 + 4)^2 - 4^2 + 9$

simplify: $(x^2 + 4)^2 - 16 + 9$

simplify: $\mathbf{(x + 4)^2 - 7}$

In general,

$$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$$

Complete the square for $x^2 + 12x - 5$.

complete the square:	$(x + 6)^2 - 6^2 - 5$
simplify:	$(x + 6)^2 - 36 - 5$
simplify:	$\mathbf{(x + 6)^2 - 41}$

Compare this to
$(x + 6)^2$
$= x^2 + 12x + 36$

Complete the square for $x^2 - 5x + 16$.

complete the square:	$(x - \frac{5}{2})^2 - (-\frac{5}{2})^2 + 16$
distributive property:	$(x - \frac{5}{2})^2 - \frac{25}{4} + 16$
simplify:	$\mathbf{(x^2 - \frac{5}{2}) + \frac{39}{4}}$

Compare this to
$(x - 2.5)^2$
$= x^2 - 5x + 6.25$

Adjust the variables using the blue arrows.

Complete the square for $x^2 + 5x + 28$.

$x^2 + 5x + 28 = (x + 0)^2 - 0 + 28$

Practice questions: completing the square

1.	Complete the square for: $x^2 + 8x$	? W
2.	Complete the square for: $x^2 + 6x - 5$	? W
3.	Complete the square for: $x^2 + x + 1$	? W
4.	Complete the square for: $x^2 - 4x + 15$	? W
5.	Complete the square for: $x^2 - 3x - \frac{3}{4}$	? W

If the coefficient of x^2 is not 1

Equations in the form $ax^2 + bx + c$ can be written in completed square form as $a(x + p)^2 + q$ by:

- first taking out the factor a from the terms containing x
- completing the square for the expression in parentheses.

For example, complete the square for $2x^2 – 4x + 1$.

First take the factor 2 out of the terms containing x:

$$2x^2 – 4x + 1 = 2(x^2 – 2x) + 1$$

Now complete the square within the parentheses:

complete the square: $= 2((x – 1)^2 – 1) + 1$

distributive property: $= 2(x – 1)^2 – 2 + 1$

simplify: $= \mathbf{2(x – 1)^2 – 1}$

Complete the square for $2x^2 + 8x + 3$.

Start by factoring the first two terms by taking out 2, which is the coefficient of x^2 in the expression:

$$2x^2 + 8x + 3 = 2(x^2 + 4x) + 3$$

$$x^2 + 4x = (x + 2)^2 - 4$$

complete the square: $= 2((x + 2)^2 - 4) + 3$

distributive property: $= 2(x + 2)^2 - 8 + 3$

simplify: $= \mathbf{2(x + 2)^2 - 5}$

Complete the square for $5 + 6x - 3x^2$.

Start by factoring the the terms containing x by –3.

$$5 + 6x - 3x^2 = 5 - 3(-2x + x^2)$$

$$= 5 - 3(x^2 - 2x)$$

$$x^2 - 2x = (x - 1)^2 - 1$$

complete the square: $= 5 - 3((x - 1)^2 - 1)$

distributive property: $= 5 - 3(x - 1)^2 + 3$

simplify: $= \mathbf{8 - 3(x - 1)^2}$

Adjust the variables using the blue arrows.

Complete the square for $3x^2 + 6x - 28$.

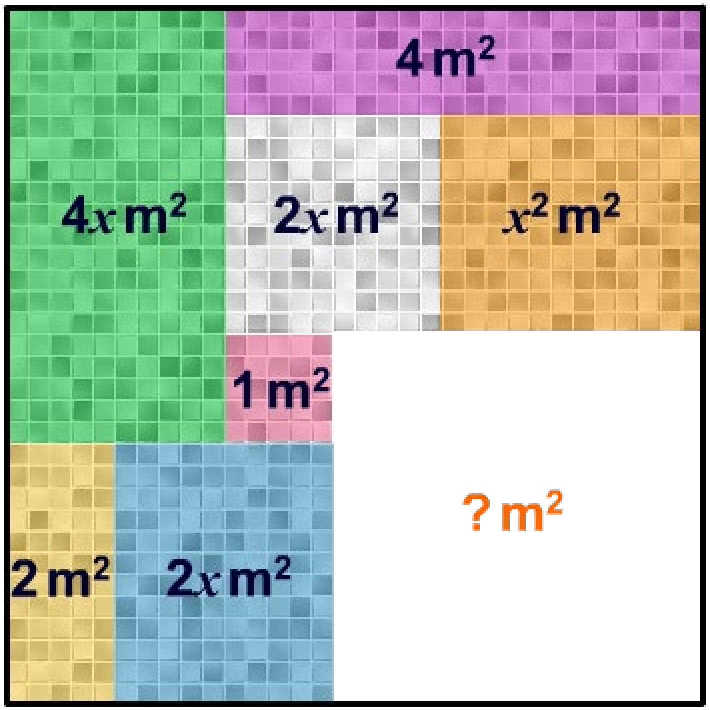
$$3x^2 + 6x - 28 = 0\,(x^2 + 0\,x) - 28$$

Practice questions: completing the square

1.	Complete the square for: $2x^2 + 8x - 5$	?
2.	Complete the square for: $5x^2 - 10x - 1$	?
3.	Complete the square for: $3x^2 - 12x$	?
4.	Complete the square for: $10 - 3x^2 - 3x$	?
5.	Complete the square for: $4x^2 + 10x - \frac{1}{2}$	?

Tiled floor

The diagram below (not to scale) shows a square floor, (x + 4) m wide, tiled in colored sections. Can you use completing the square to find the un-tiled area?



The tiled area is a perfect square, minus the un-tiled area.

Sum the tiled areas:

$4x + 4 + 2x + x^2 + 2 + 2x + 1$

$= x^2 + 8x + 7$

Complete the square to find the subtracted (un-tiled) area:

$x^2 + 8x + 7 = (x + 4)^2 - 16 + 7$

$= (x + 4)^2 - 9$

The un-tiled area measures **9 m²**.

Solving quadratic equations

Quadratic equations that cannot be solved by factoring can be solved by completing the square.

For example, the quadratic equation $x^2 - 4x - 3 = 0$ can be solved by completing the square as follows:

$$(x - 2)^2 - 4 - 3 = 0$$

simplify: $(x - 2)^2 - 7 = 0$

add 7 to both sides: $(x - 2)^2 = 7$

How would you complete the solution?

square root both sides: $x - 2 = \pm\sqrt{7}$

$\mathbf{x = 2 + \sqrt{7}}$ or $\mathbf{x = 2 - \sqrt{7}}$

Solve $x^2 + 8x + 5 = 0$ by completing the square, writing the answer as a decimal to the nearest thousandth.

Complete the square on the left-hand side:

$$(x + 4)^2 – 16 + 5 = 0$$

simplify: $(x + 4)^2 – 11 = 0$

add 11 to both sides: $(x + 4)^2 = 11$

square root both sides: $x + 4 = \pm\sqrt{11}$

$x = –4 + \sqrt{11}$ or $x = –4 – \sqrt{11}$

$\mathbf{x = –0.683}$ $\mathbf{x = –7.317}$

(to nearest thousandth)

Solve $3x^2 - 6x - 5 = 0$ by completing the square.

take out the factor 3 from x terms: $3(x^2 - 2x) - 5 = 0$

complete the square: $3((x - 1)^2 - 1) - 5 = 0$

distributive property $3(x - 1)^2 - 3 - 5 = 0$

$3(x - 1)^2 - 8 = 0$

add 8 to both sides: $3(x - 1)^2 = 8$

divide both sides by 3: $(x - 1)^2 = \frac{8}{3}$

square root both sides: $x - 1 = \pm\sqrt{\frac{8}{3}}$

$$\boldsymbol{x = 1 + \sqrt{\frac{8}{3}}} \quad \text{or} \quad \boldsymbol{x = 1 - \sqrt{\frac{8}{3}}}$$