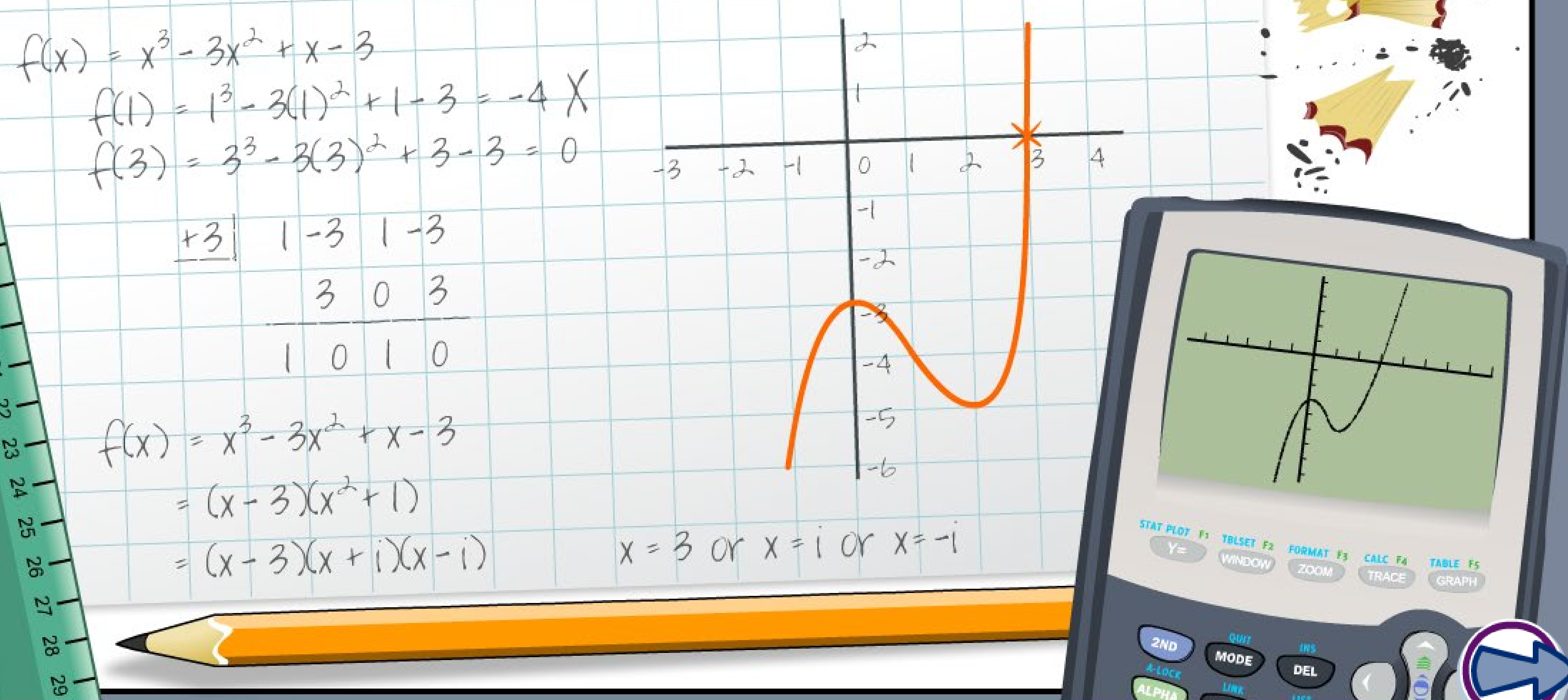
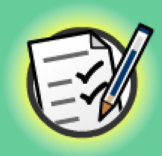
Binomial Theorem



Common core icons



This icon indicates a slide where the Standards for Mathematical Practice are being developed. Details of these are given in the Notes field.

Slides containing examples of mathematical modeling are marked with this stamp.

This icon indicates an opportunity for discussion or group work.

The **Standards for Mathematical Practice** outlined in the Common Core State Standards for Mathematics describe varieties of expertise that mathematics educators at all levels should seek to develop in their students.

These are:

1) Make sense of problems and persevere in solving them.

2) Reason abstractly and quantitatively.

3) Construct viable arguments and critique the reasoning of others.

4) Model with mathematics.

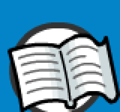
5) Use appropriate tools strategically.

6) Attend to precision.

7) Look for and make use of structure.

8) Look for and express regularity in repeated reasoning.

This icon indicates that the slide contains activities created in Flash. These activities are not editable.



This icon indicates teacher's notes in the Notes field.

Binomial expressions

A **binomial expression** is an expression containing two terms.

Examples: $x^2 + 4$, $y^7 - 15x$, $3a + 5b$, $5x^5 - 3x^4$, $2p^2 + 2q^2$…

Write a for the first term and b for the second term, then $a + b$ can represent any binomial expression.

Expand the powers of binomial expressions below.

$(a + b)^0 = 1$

$(a + b)^1 = 1a + 1b$

$(a + b)^2 = 1a^2 + 2ab + 1b^2$

$(a + b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$

$(a + b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$

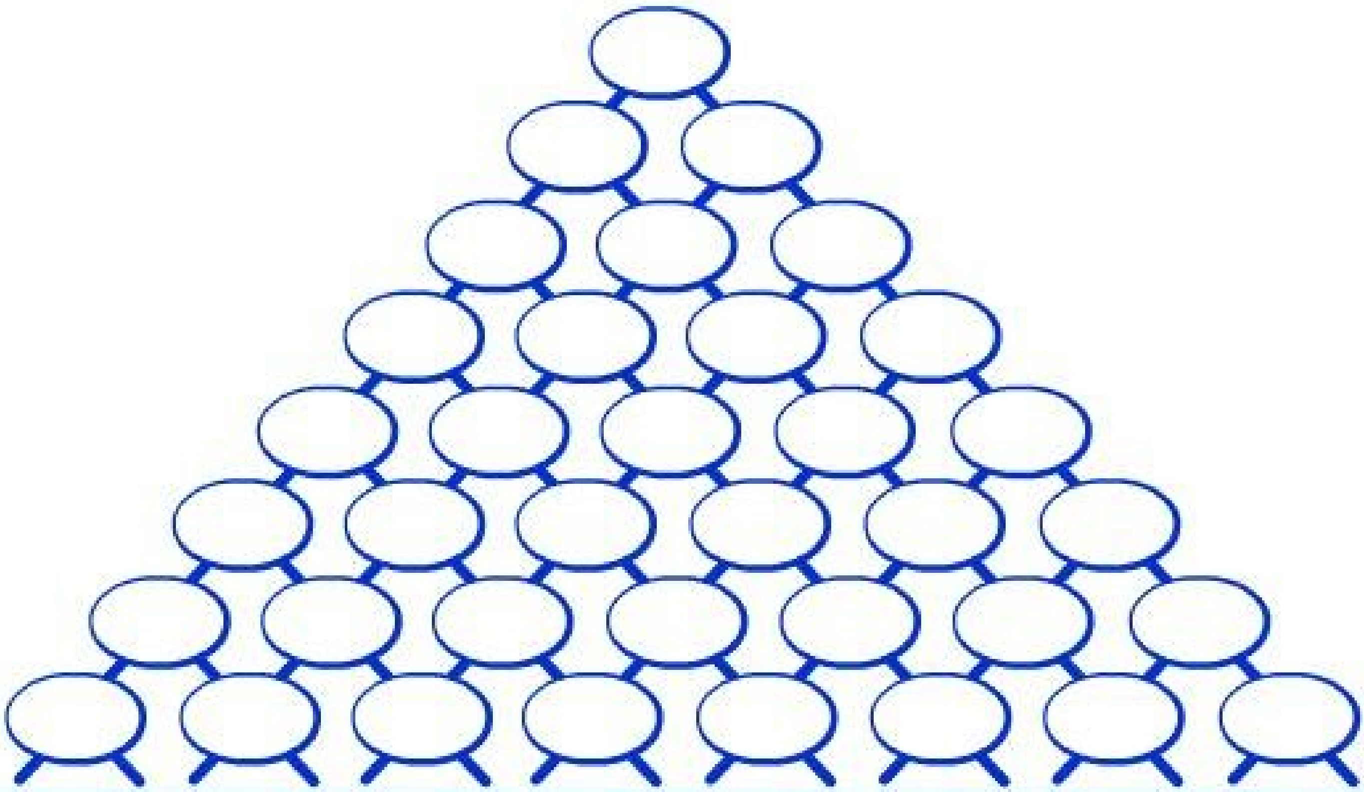
$(a + b)^5 = 1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5$

Can you spot a pattern linking the coefficients?

Pascal’s triangle

Can you predict the values in Pascal’s triangle?

Press the ovals to reveal the values.



Binomial theorem

$(a + b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

Notice that:

- powers of a start at n and decrease to 0
- powers of b start at 0 and increase to n
- the sum of the powers of a and b is n for each term
- altogether, there are $n + 1$ terms
- and the coefficients match the fifth row of Pascal's triangle.

Binomial theorem: the coefficients in the expansion of $(a + b)^n$ are given by the $(n + 1)^{\text{th}}$ row of Pascal's triangle.

Using Pascal's triangle

Expand $(a + b)^6$.

calculate the seventh row of Pascal's triangle: 1 6 15 20 15 6 1

find the powers in each term:

$$(a + b)^6 = \mathbf{a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6}$$

Expand $(x + 1)^5$.

calculate the sixth row of Pascal's triangle: 1 5 10 10 5 1

find the powers in each term:

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

replace a with x and b with 1:

$$(x + 1)^5 = \mathbf{x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1}$$

Negative terms

Expand $(2x - y)^4$.

calculate the fifth row of Pascal's triangle: 1 4 6 4 1

find the powers in each term:

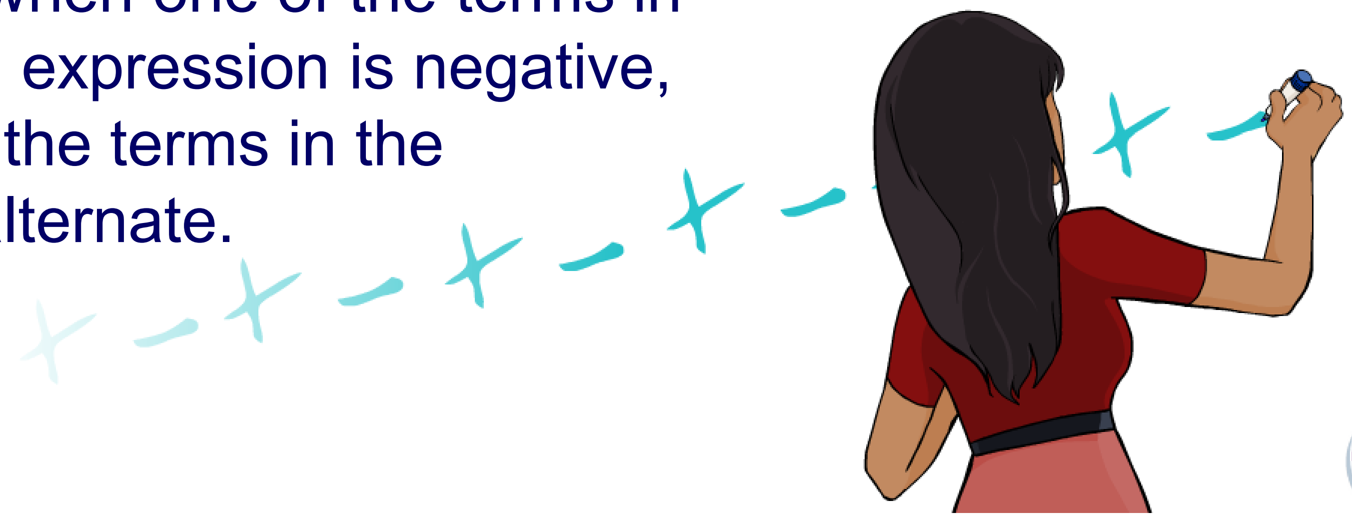
$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

replace a with $2x$ and b with $-y$:

$$(2x - y)^4 = (2x)^4 + 4(2x)^3(-y) + 6(2x)^2(-y)^2 + 4(2x)(-y)^3 + (-y)^4$$

$$= \mathbf{16x^4 - 32x^3y + 24x^2y^2 - 8xy^3 + y^4}$$

Notice that when one of the terms in the binomial expression is negative, the signs of the terms in the expansion alternate.



Binomial expansions

1. Expand $(3x + 7)^4$. ?
2. Expand $(2 - x)^5$. ?
3. Expand $(6x + 2)^3$. ?
4. Expand $(x - 12)^3$. ?
5. Expand $(-2x - 2)^7$. ?

Finding binomial coefficients

What do the coefficients in a binomial expansion mean?

Complete the table below for the expansion of $(a + b)^4$.

$$(a + b)^4 = (a + b)(a + b)(a + b)(a + b)$$

ways to get a^4	ways to get a^3b	ways to get a^2b^2	ways to get ab^3	ways to get b^4
aaaa	*aaab* *aaba* *abaa* *baaa*	*aabb* *abab* *abba* *bbaa* *baba* *baab*	*abbb* *babb* *bbab* *bbba*	*bbbb*
1 way	**4** ways	**6** ways	**4** ways	**1** way

Combination notation

In combination theory, the number of ways of getting r objects from a total number of objects, n, is written ${}_nC_r$. This is read "n choose r". It can also be written $\binom{n}{r}$.

For example, the number of ways of getting exactly one b in a term when expanding $(a + b)^4$ is ${}_4C_1$ or $\binom{4}{1}$. The rest of the term will contain three a's, so this is the same as ${}_4C_3$ or $\binom{4}{3}$.

Write the coefficients in the binomial expansion of $(a + b)^4$ using this notation. *Press the buttons to see how.*

${}_nC_r$ $\binom{n}{r}$ =

The number of ways to choose r objects from a group of n objects is written as ${}_nC_r$ and is given by

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

$n!$ is read as "**n factorial**" and is the product of all the natural numbers from 1 to n. For $n = 0$, by definition $0! = 1$.

In general: $n! = n \times (n-1) \times (n-2) \times (n-3) \times \ldots \times 2 \times 1$

Evaluate: a) 5! b) 12! c) 20!

$5! = 5 \times 4 \times 3 \times 2 \times 1 =$ **120**

$12! = 12 \times 11 \times 10 \times \ldots \times 2 \times 1 =$ **479,001,600**

$20! = 20 \times 19 \times 18 \times \ldots \times 2 \times 1 =$ **2,432,902,008,176,640,000**

Canceling

Evaluate $_4C_2$ using the formula: $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$

$$_4C_2 = \binom{4}{2} = \frac{4!}{2!2!} = \frac{4 \times 3 \times \cancel{2 \times 1}}{(\cancel{2 \times 1})(2 \times 1)} = \frac{\overset{2}{\cancel{4}} \times 3}{\cancel{2} \times 1} = \mathbf{6}$$

Notice that many of the numbers cancel out.

The effect of this canceling gives alternative forms for $_nC_r$:

$$\binom{n}{r} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!} = \frac{n(n-1)(n-2)\ldots(r+1)}{(n-r)!}$$

The binomial theorem: $(a + b)^n$ can be written as:

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{n}b^n$$

where $$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

A special case is $(1 + x)^n$:

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots + x^n$$

$$= 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + x^n$$

Using the binomial theorem

Find the coefficient of a^7b^3 in the expansion of $(a - 2b)^{10}$.

binomial theorem: $\binom{10}{3} a^7(-2b)^3 = \frac{10 \times \overset{3}{\cancel{9}} \times \overset{4}{\cancel{8}}}{\cancel{3} \times \cancel{2} \times 1} a^7(-8b^3)$

cancel: $= 120(-8a^7b^3)$

$= -960a^7b^3$

So the coefficient of a^7b^3 in the expansion of $(a - 2b)^{10}$ is **–960**.

Find the coefficient of x^5 in the expansion of $(3x + 4)^7$.

binomial theorem: $\binom{7}{5} (3x)^5(4)^2 = \frac{7 \times \overset{3}{\cancel{6}}}{\cancel{2} \times 1} (3)^5(4)^2x^5$

cancel: $= 21(3888x^5)$

$= 81648x^5$

So the coefficient of x^5 in the expansion of $(3x + 4)^7$ is **81648**.

Use the *binomial theorem* to write down the first four terms in the multiplication of $(1 + x)^7$ in ascending powers of x.

$$(1 + x)^7 = \binom{7}{0} + \binom{7}{1}x + \binom{7}{2}x^2 + \binom{7}{3}x^3 + \ldots$$

$$= 1 + 7x + \frac{7 \times \cancel{6}^{\,3}}{\cancel{2} \times 1}x^2 + \frac{7 \times \cancel{6} \times 5}{\cancel{3} \times \cancel{2} \times 1}x^3 + \ldots$$

$$= \mathbf{1 + 7x + 21x^2 + 35x^3 + \ldots}$$

How could we use this to find an approximate value for 1.1^7?

How could we use the *binomial theorem* to find an approximate value for 1.1^7?

Write 1.1 as a binomial expression by letting $x = 0.1$, then $1.1 = (1 + x)$.

Then $1.1^7 = (1 + x)^7$

binomial theorem: $(1 + x)^7 = 1 + 7x + 21x^2 + 35x^3 + \ldots$

evaluate: $1.1^7 \approx 1 + 7 \times 0.1 + 21 \times 0.1^2 + 35 \times 0.1^3$

As 0.1 is raised to ever higher powers it becomes much smaller and so less significant. We can therefore leave out higher powers of x and still have a reasonable approximation.

$$1.1^7 \approx 1 + 0.7 + 0.21 + 0.035$$

$$\approx \mathbf{1.945}$$