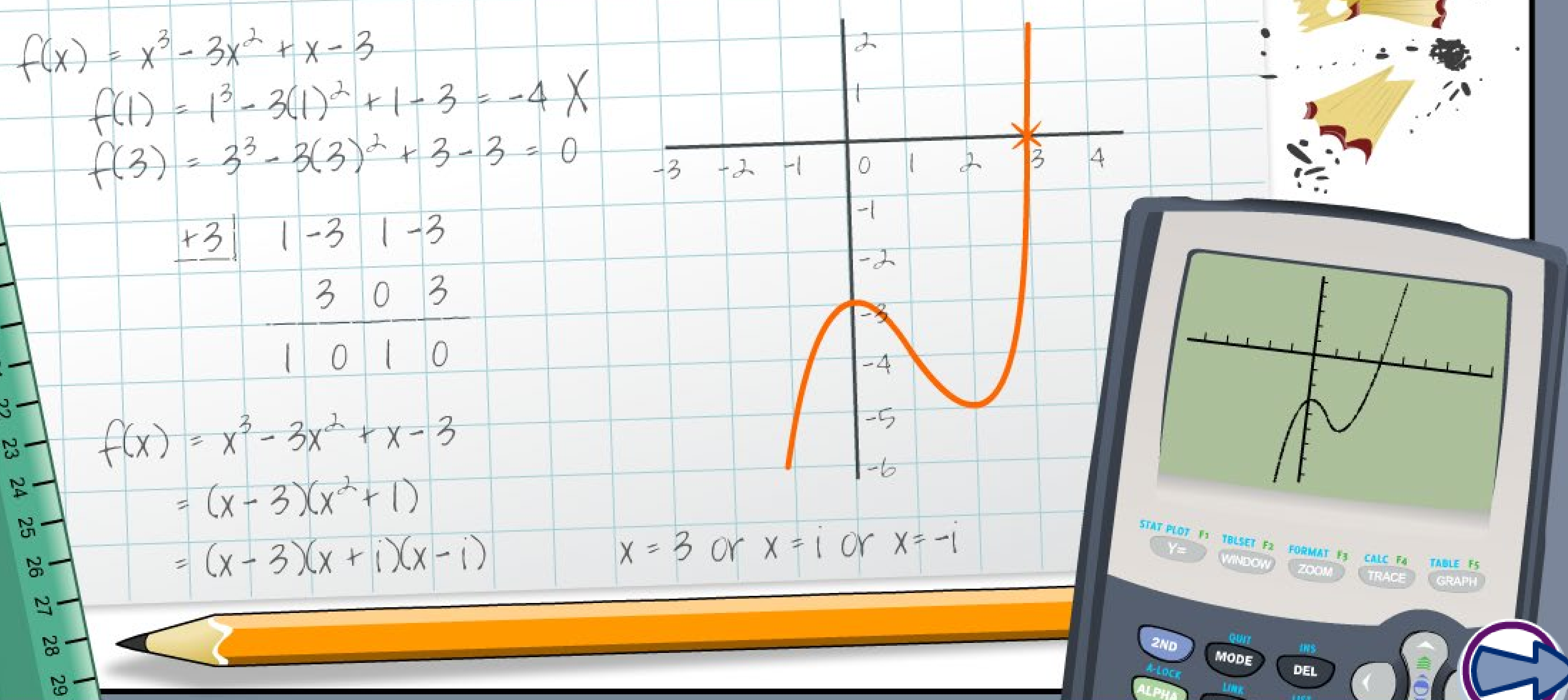
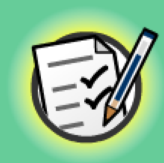
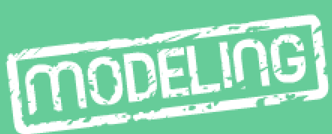
Exponential and Logarithmic Equations



Common core icons



This icon indicates a slide where the Standards for Mathematical Practice are being developed. Details of these are given in the Notes field.



Slides containing examples of mathematical modeling are marked with this stamp.

This icon indicates an opportunity for discussion or group work.

The **Standards for Mathematical Practice** outlined in the Common Core State Standards for Mathematics describe varieties of expertise that mathematics educators at all levels should seek to develop in their students.

These are:

1) **Make sense of problems and persevere in solving them.**
2) **Reason abstractly and quantitatively.**
3) **Construct viable arguments and critique the reasoning of others.**
4) **Model with mathematics.**
5) **Use appropriate tools strategically.**
6) **Attend to precision.**
7) **Look for and make use of structure.**
8) **Look for and express regularity in repeated reasoning.**

This icon indicates that the slide contains activities created in Flash. These activities are not editable.

This icon indicates teacher's notes in the Notes field.

Solving exponential equations

Simple **exponential equations** such as $5^x = 125$ can be solved by observation. Solution: $x = 3$, since $5^3 = 125$.

Also, exponential equations containing terms with the same base can be solved by **equating exponents**.

For example, $7^{x+3} = 7^{2x}$

equate exponents: $x + 3 = 2x$

solve: $\mathbf{3 = x}$

How would you solve the equation $5^x = 25^{x+1}$?

rewrite in base 5: $5^x = (5^2)^{x+1}$

distribute: $5^x = 5^{2x+2}$

equate exponents: $x = 2x + 2$

solve: $\mathbf{x = -2}$

How can we find the value of k in the equation $2^k = 35$?

$2^k = 35$

solve: $k = ?$

When we cannot find the value of x by observation, we need to be able to find the **inverse** of the exponential.

The inverse of an exponential is called a **logarithm**.

logarithm:
if $x = a^y$, then $y = \log_a x$

for a positive value of x and for $a > 0$, $a \neq 1$

Can you rewrite the equation $2^k = 35$ using logarithms?

Using logarithms

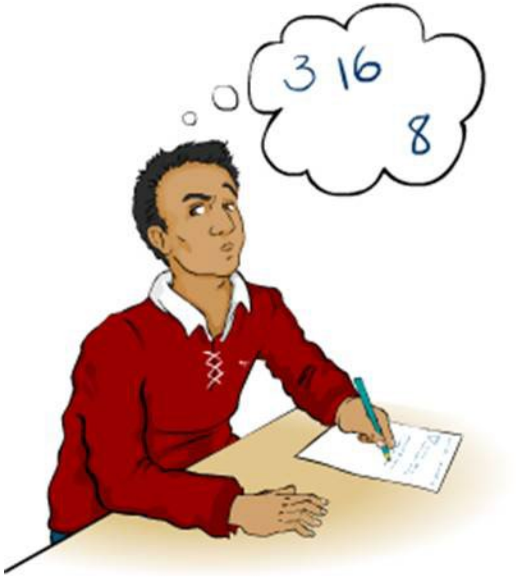
Rewrite $2^k = 35$ using the definition of a logarithm.

$$2^k = 35$$

if $x = a^y$ then $y = \log_a x$: $k = \log_2 35$

This is now an equation in k that can be solved by evaluating the logarithm base 2 of 35.

However, there is no button on a standard calculator to find logarithms base 2, only base 10 (“log”) or base e (“ln”).



To solve this equation, there are two options:

- take logarithms base 2, then change the logarithmic base to 10 or e
- take logarithms base 10 or e.

Match the equivalent logarithmic expressions

$\log_a x \,/\, \log_a b$	$\log_b (x/y)$
$\log_b b$	$\log_b x$
$\log_b (x^n)$	1
$\log_b x + \log_b y$	$n \log_b x$
$\log_b x - \log_b y$	$\log_b (xy)$

Using logarithms to solve an exponential equation

Solve the equation $2^k = 35$ to the nearest thousandth.

Method 1: take logarithms base 2 then change base to 10 or e.
Method 2: take logarithms base e or 10 first instead.

Both methods produce the same answer, however it is more common to use method 2 to solve exponential equations, because it avoids having to change base half way through the calculation.

Press start to work through each method. start

Solving logarithmic equations

Some **logarithmic equations** can be solved by observation.

For example, solve: $\log x = 2$.

Since $10^2 = 100$, $x = 100$.

Logarithmic equations containing terms with the same logarithmic base can be solved by **equating arguments**.

Solve the equation $\ln(x + 1) = \ln 9$.

$\ln(x + 1) = \ln 9$

equate arguments: $x + 1 = 9$

solve: $\mathbf{x = 8}$

How would you solve the equation $\ln x = \log_2 5$?

How would you solve the equation $\ln x = \log_2 5$?

Write both terms in the same logarithmic base, then equate arguments.

$$\ln x = \log_2 5$$

change right side to base e: $\ln x = \dfrac{\ln 5}{\ln 2}$

evaluate: $\ln x = 2.3219$

by definition of natural logarithm: $x = e^{2.3219}$

evaluate: **x = 10.20 to nearest hundredth**

Write the constant value 3 in the form $\log_4 x$.

since $\log_4 4 = 1$: $3 = 3(\log_4 4)$

exponent rule: $3 = \log_4 (4^3)$

simplify: $3 = \mathbf{\log_4 64}$

Solving logarithmic equations

Solve $\log_5 x + 2 = \log_5 10$.

To solve this equation, all the terms must be logarithms in the same base.

First, write the constant value 2 in logarithmic form.

since $\log_5 5 = 1$: $2 = 2\log_5 5$

exponent rule: $2 = \log_5 (5^2)$

simplify: $2 = \log_5 25$

The equation can now be written as:

$$\log_5 x + \log_5 25 = \log_5 10$$

product rule: $\log_5 (25x) = \log_5 10$

equate arguments: $25x = 10$

evaluate: $\mathbf{x = 0.4}$

Be careful!

Remember, the argument of a logarithm (i.e. the "x" in log x) must be a **positive number**.

Always check your solution in the original logarithmic expression(s) to make sure it is valid.

For which values of x is the expression log (x + 1) valid?

The argument of the logarithm must be positive.

$$x + 1 > 0$$

$$x > -1$$

The expression is valid for any values of x greater than –1.

Exponential and logarithmic equations

Solve...

1) $5^{2x} = 30$ to the nearest hundredth ?

2) $4^{3x+1} = 7^{x+2}$ to the nearest hundredth ?

3) $3 - \log_2 x = \log_2 4$?

4) $3^{2x} - 5(3^x) + 4 = 0$ to the nearest hundredth ?

5) $\log_5 x + \log_5 (x + 3) = \log_5 4$?

Savings account

Hayley puts $1000 into a new savings account with a 2.7% monthly compound interest rate.

Write a formula for A, the amount in her account, after n months of saving.

Use your formula to calculate:
a) the amount Hayley will have after 2 years
b) the number of months it will take for the account to contain $1500.

a) The rate is 2.7%, so for every month, n, the amount will be multiplied by 1.027 (starting with the original amount, $1000.)

A function to describe the amount, A, after n months is:

$$A = 1000(1.027^n)$$

Savings account

The amount, A, after n months is:

$$A = 1000(1.027^n)$$

To find the number of months it will take for the account to contain $1500, substitute A = 1500 into the equation.

substitute A = 1500:	$1500 = 1000(1.027^n)$
divide by 1000:	$1.5 = 1.027^n$
take logarithms:	$\ln 1.5 = \ln (1.027^n)$
exponent rule:	$\ln 1.5 = n \ln 1.027$
rearrange:	$n = \ln 1.5 / \ln 1.027$
evaluate:	$n = 15.2 \approx$ **16 months to the nearest whole month**

The pH scale

The pH scale

In chemistry, a solution's pH is defined by the logarithmic equation **$pH = -\log H^+$**, where H^+ is the solution's hydrogen ion concentration in moles per liter.

pH values are usually rounded to the nearest tenth.

Press **start** to begin a multiple choice quiz using this equation.

start

Bacterial decay

A man has a bacterial infection. There are 100,000 bacteria present, but when an antibiotic is introduced, the number of bacteria is reduced by half every 2 hours.

a) Write a function to model the number of bacteria after h hours.
b) A person is considered "cured" of this infection when fewer than 1000 bacteria are present. How long, to the nearest hour, until the man is cured?

a) After 2 hours, there are 100,000(0.5).
After 4 hours, there are $100{,}000(0.5^2)$.
After 6 hours, there are $100{,}000(0.5^3)$.

A function to model the number of bacteria (y) after h hours is: $y = 100{,}000(0.5^{h/2})$

A function to model the number of bacteria (y) after h hours is:

$$y = 100{,}000(0.5^{h/2})$$

b) To find the number of hours until the man is "cured" (i.e. there are fewer than 1000 bacteria present), substitute y = 1000 into the function.

substitute y = 1000:	$1000 = 100{,}000(0.5^{h/2})$
divide by 100,000:	$0.01 = 0.5^{h/2}$
take logarithms:	$\ln 0.01 = \ln (0.5^{h/2})$
exponent rule:	$\ln 0.01 = (h / 2) \ln 0.5$
rearrange:	$h = 2(\ln 0.01 / \ln 0.5)$
evaluate:	$h = 13.3 \approx$ **14 hours** to the nearest hour