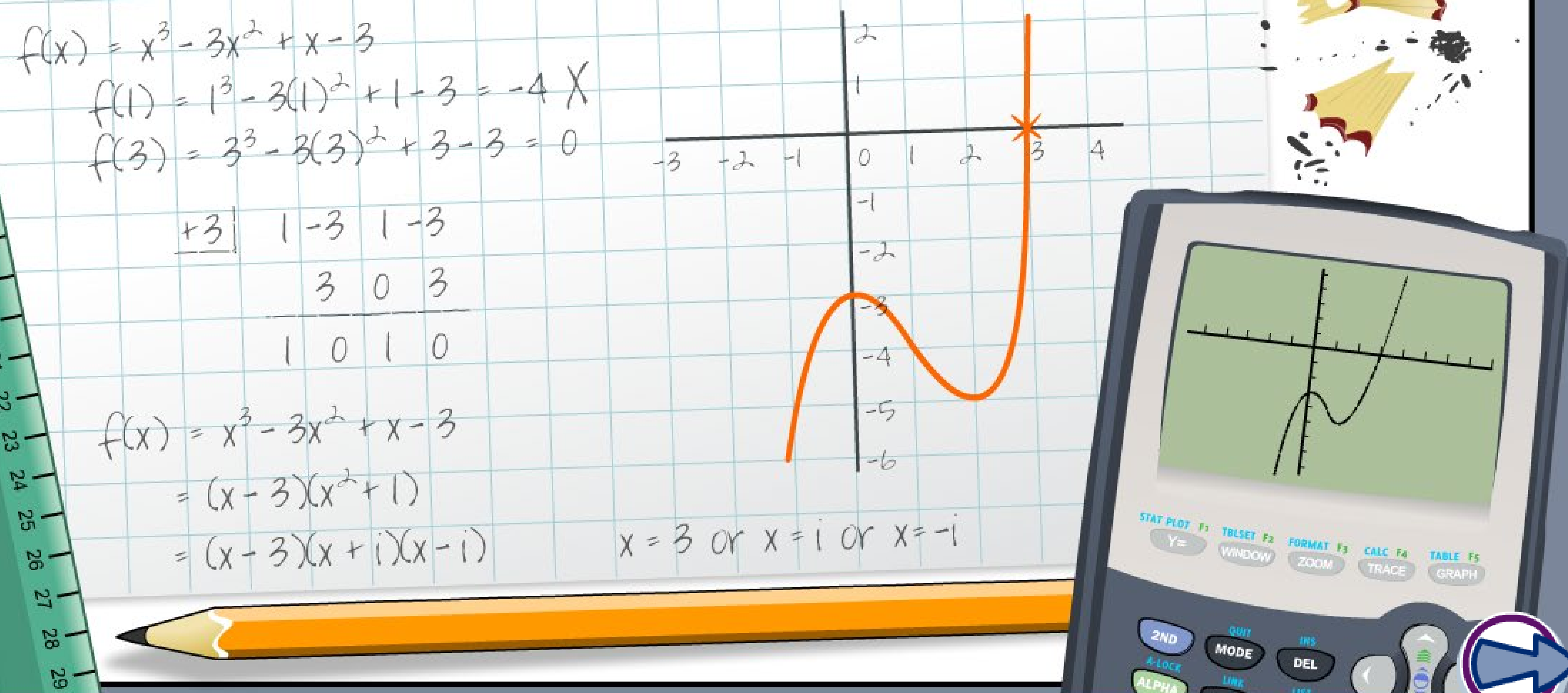
Solving Polynomial Equations



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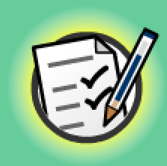
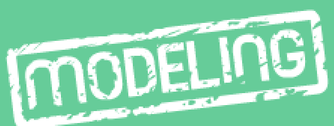
Information

Common core icons



This icon indicates a slide where the Standards for Mathematical Practice are being developed. Details of these are given in the Notes field.



Slides containing examples of mathematical modeling are marked with this stamp.

This icon indicates an opportunity for discussion or group work.

The **Standards for Mathematical Practice** outlined in the Common Core State Standards for Mathematics describe varieties of expertise that mathematics educators at all levels should seek to develop in their students.

These are:

1) Make sense of problems and persevere in solving them.

2) Reason abstractly and quantitatively.

3) Construct viable arguments and critique the reasoning of others.

4) Model with mathematics.

5) Use appropriate tools strategically.

6) Attend to precision.

7) Look for and make use of structure.

8) Look for and express regularity in repeated reasoning.

This icon indicates that the slide contains activities created in Flash. These activities are not editable.

This icon indicates teacher's notes in the Notes field.

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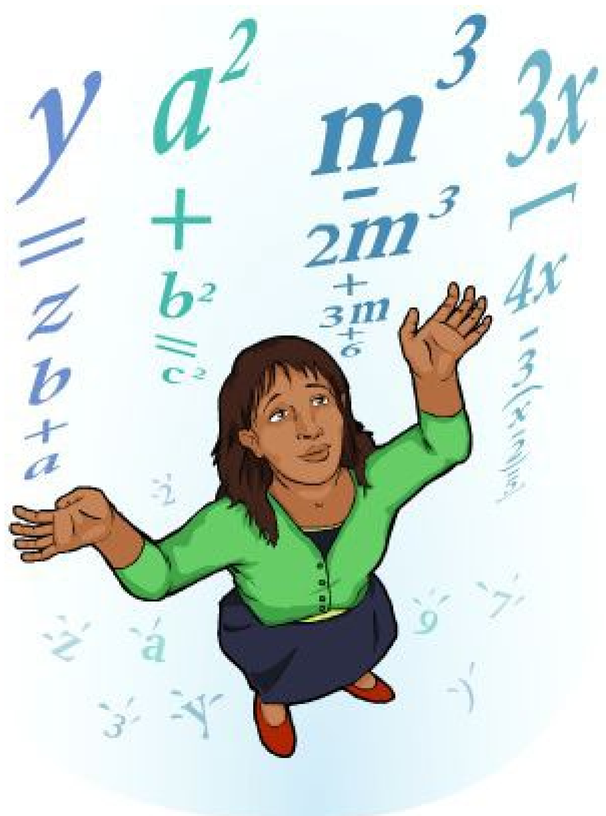
Polynomial equations

A **polynomial equation** is an equation where one polynomial expression is set equal to another polynomial expression.

For example, $3x^4 + 2x^2 - 7 = x^4 + x + 1$



Polynomial equations are solved by finding the roots of the equation, i.e. the values of x that make the equation true.

They can be solved using techniques like **factoring**, **graphing**, **completing the square** and the **quadratic formula**.

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The fundamental theorem of algebra

The fundamental theorem of algebra (***FTA***):
every polynomial of degree n has exactly n roots, including multiple and complex roots.

How many roots does each equation have?
Use a graph to find out how many roots are real and how many are imaginary.

a) $x^2 - 4 = 0$	2 roots (both real)
b) $2x + 7 = 13$	1 root (real)
c) $6x^3 - 32x^2 - 24x + 64 = 0$	3 roots (real)
d) $3x^4 - 21x^2 - 24 = 0$	4 roots (2 real, 2 imaginary)

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Solving an equation given a root

Solve the equation $x^3 - 4x^2 - 11x + 30 = 0$ given that one of its roots is $x = 5$.

FTA: The equation has three roots, so there are two to find.

$x = 5$ is a root, so $(x - 5)$ is a factor.

divide by $(x - 5)$ to find the quotient:

$$\begin{array}{c|cccc} 5 & 1 & -4 & -11 & 30 \\ & & 5 & 5 & -30 \\ \hline & 1 & 1 & -6 & 0 \end{array}$$

$(x^2 + x - 6)$ is the quotient.

$(x - 5)(x^2 + x - 6) = 0$

factor: $(x - 5)(x - 2)(x + 3) = 0$

zero product property: $x - 5 = 0$ or $x - 2 = 0$ or $x + 3 = 0$

solve: $\mathbf{x = 5}$ **or** $\mathbf{x = 2}$ **or** $\mathbf{x = -3}$

Confirm the results graphically.

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Find the roots of the following polynomial equations:

a) $x^2 + 5x + 6 = 0$

b) $x^2 - 6x + 9 = 0$

	a) $x^2 + 5x + 6 = 0$		**b)** $x^2 - 6x + 9 = 0$	
factor:	$(x + 2)(x + 3) = 0$		$(x - 3)(x - 3) = 0$	
zero product property:	$x + 2 = 0$ or $x + 3 = 0$		$x - 3 = 0$ or $x - 3 = 0$	
solve:	$x = -2$	or $x = -3$	$x = 3$	or $x = 3$
solution set:	**{–2, –3}**		**{3}**	

The first equation has two different roots that each occur once. –2 and –3 are roots with **multiplicity** one.

The second equation has the same root occurring twice. The **multiplicity** of the root 3 is two.

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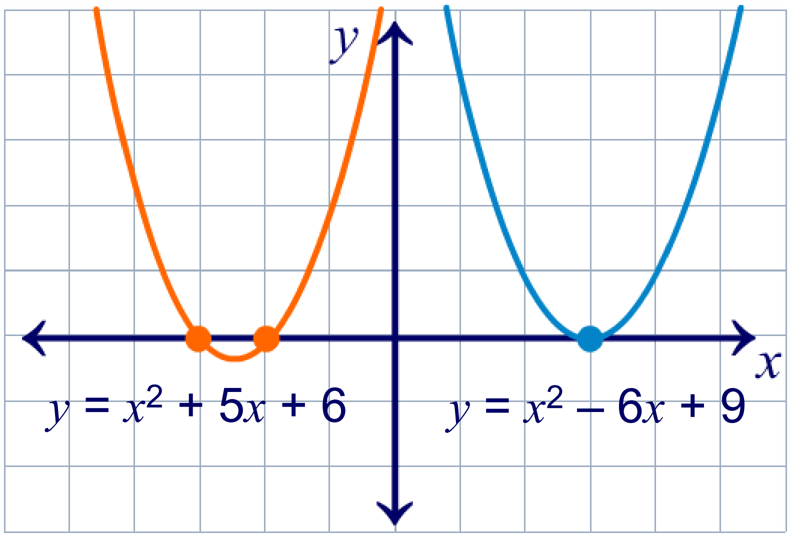
Graphs and multiplicity

Describe how the graphs of $y = x^2 + 5x + 6$ and $y = x^2 - 6x + 9$ behave at the roots.



$y = x^2 + 5x + 6$ crosses the x-axis at $x = -2$ and $x = -3$, both roots with multiplicity one.

$y = x^2 - 6x + 9$ is tangent to the x-axis at $x = 3$, a root with multiplicity two.

If the multiplicity of a real root is **odd**, the graph **crosses** the x-axis.

If the multiplicity of a real root is **even**, the graph is **tangent** to the x-axis.

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Use the quadratic formula to solve the equation $x^2 + x + 1 = 0$, then graph $y = x^2 + x + 1$. What kind of solutions does it have? Does it cross the x-axis?

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1 - 4}}{2} = -\tfrac{1}{2} \pm \tfrac{\sqrt{3}}{2}i$$

The equation has two imaginary roots that both have multiplicity one.

The graph $y = x^2 + x + 1$ does not touch the x-axis.

$-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i$ and $-\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i$ are **complex conjugates**.

Complex conjugate theorem: if $P(x)$ is a polynomial expression with real coefficients, then $a + bi$ is a root of $P(x) = 0$ if and only if $a - bi$ is also a root.

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True or false

Are these statements about polynomials true or false?

1. If a polynomial equation $P(x) = 0$ has odd degree, then it has no real roots. ?
2. There is a cubic equation with solutions at $x = -3$ (multiplicity two), $x = -1$, and $x = 4$ (multiplicity two). ?
3. If the graph of $y = P(x)$ is tangent to the x-axis, then the equation $P(x) = 0$ has an imaginary root. ?
4. An equation with degree 5 has a total of 5 roots when multiplicity is taken into consideration. ?
5. $P(x) = 0$ must have an even number of imaginary roots $(a \pm bi)$, $b \neq 0$. ?

true false

S C ?

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Given an imaginary root

Solve $P(x) = x^4 - 4x^3 - 44x^2 + 196x - 245 = 0$ given that one of its roots is $x = 2 + i$.

FTA: The equation has four roots, so there are three to find.

complex conjugate theorem: Since $(2 + i)$ is a root, $(2 - i)$ must also be a root.

$$(x - (2 + i))(x - (2 - i)) = (x^2 - 4x + 5)$$

divide by comparing coefficients:

$$P(x) = (x^2 - 4x + 5)(ax^2 + bx + c)$$

$$= ax^4 + (b - 4a)x^3 + (5a + c - 4b)x^2 + (5b - 4c)x + 5c$$

by inspection: $a = 1$, $b = 0$, $c = -49$

substitute: $(x - (2 + i))(x - (2 - i))(x^2 - 49) = 0$

factor by difference of squares: $(x - (2 + i))(x - (2 - i))(x - 7)(x + 7) = 0$

zero product property: $\{(2 + i), (2 - i), 7, -7\}$

Confirm the results graphically.

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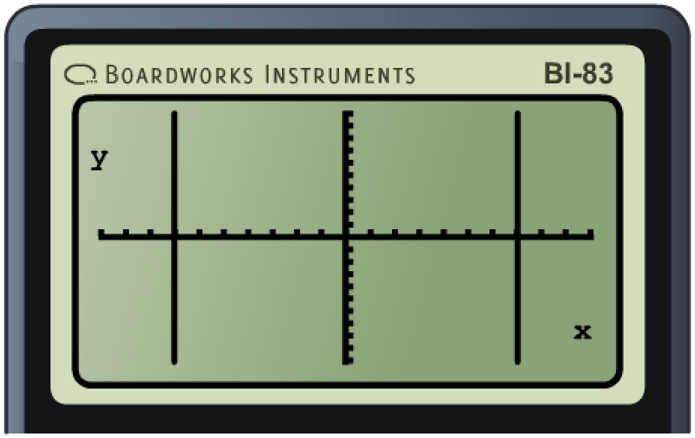
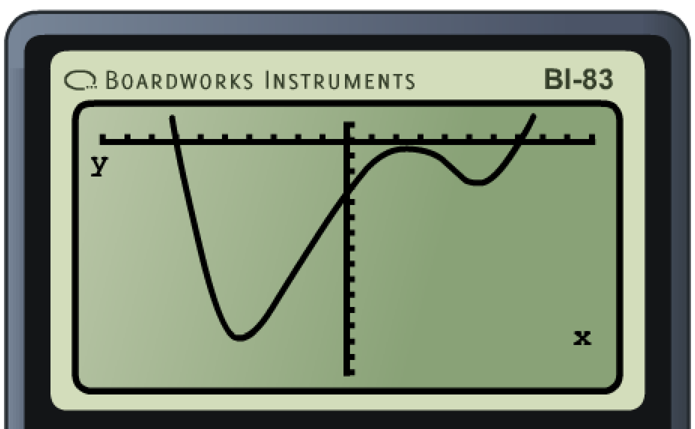
Press "Y=" on a graphing calculator and enter $y = x^4 - 4x^3 - 44x^2 + 196x - 245$. Press "GRAPH" to view the graph.



window: [–10, 10] for x, [–10, 10] for y



window: [–10, 10] for x, [–1500, 100] for y

The graph intercepts the x-axis at $x = -7$ and $x = 7$.

Change the window to confirm that the graph does not cross the axis anywhere else.

This means the polynomial equation has two real roots.

By the ***fundamental theorem of algebra***, it must therefore have two imaginary roots.

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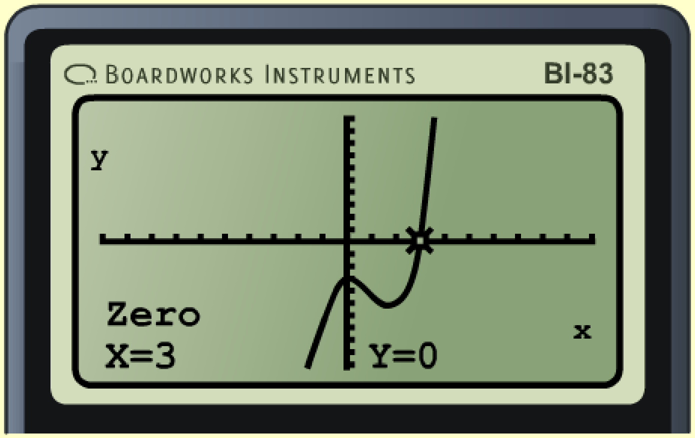
Using a graph to find roots (1)

Graphically determine the real roots of $x^3 - 3x^2 + x - 3 = 0$. Then factor to solve the equation.



Use the "CALC" feature on a graphing calculator to find the zeros.

The graph has one real root at $x = 3$.

$(x - 3)$ is therefore a factor.

By ***FTA*** there are three roots, so there are two imaginary roots.

divide by synthetic division:

3	1	–3	1	–3
		3	0	3
	1	0	1	0

$$(x - 3)(x^2 + 1) = 0$$

factor sum of squares: $(x - 3)(x + i)(x - i) = 0$

zero product property: $\mathbf{x = 3 \text{ or } x = -i \text{ or } x = i}$

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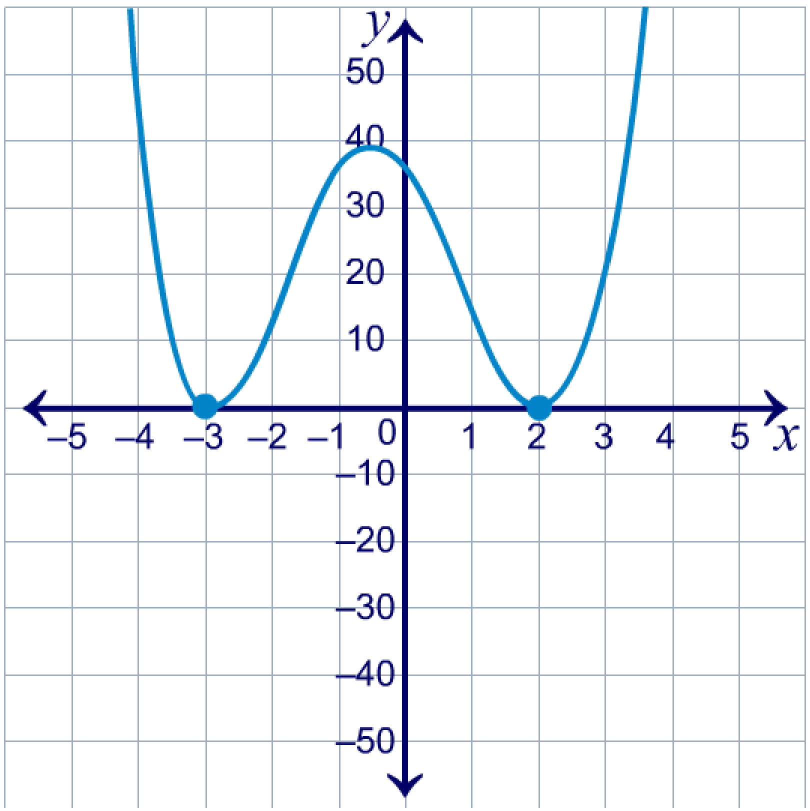
Use the graph of $P(x) = x^4 + 2x^3 - 11x^2 - 12x + 36$ to determine the roots of $P(x) = 0$ and state their multiplicity.



There are roots at $x = 2$ and $x = -3$.

The graph is tangent to the x-axis at both roots, so they have **even** multiplicity.

By the fundamental theorem of algebra, there are 4 roots.

Therefore both roots have multiplicity 2.

The solution set to the equation $P(x) = 0$ is **{2, –3}**.

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Write the quintic polynomial in standard form that has roots $x = 2$ (multiplicity 2), $x = -4$ (multiplicity 1), and $x = -3 \pm i$, and passes through the point (0, 320).

FTA: Quintic polynomials have 5 roots, so all of the roots are known.

If $x = c$ is a root of multiplicity n, then $(x - c)^n$ is a factor.

There may also be a constant factor, a.

write in factored form: $P(x) = a(x - 2)^2(x + 4)(x + 3 + i)(x + 3 - i)$

expand and simplify: $P(x) = a(x^5 + 6x^4 - 2x^3 - 56x^2 - 24x + 160)$

substitute (0, 320) to find a: $320 = a(160)$

$$a = \frac{320}{160} = 2$$

substitute a: $\mathbf{P(x) = 2x^5 + 12x^4 - 4x^3 - 112x^2 - 48x + 320}$

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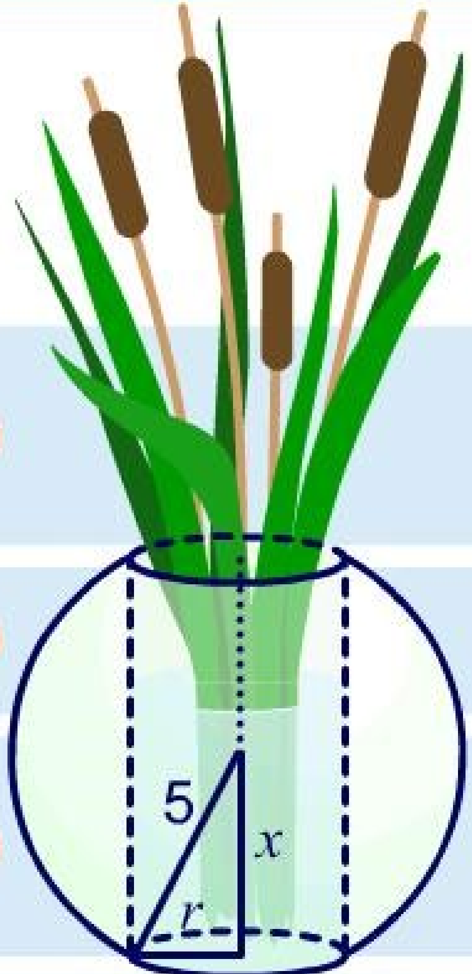
Application

A glass vase is made so that the outside forms part of a sphere with radius 5 cm while the stems sit in a cylindrical hole inscribed in the sphere. Let x be half the height of the vase.

1. **Determine a function $V(x)$ that gives the volume of the cylinder and state its domain.**
2. **Graphically determine the maximum volume of the cylinder.**
3. **If the cylinder must hold at least 200 ml, state a set of x-values that will accommodate this volume.**



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Solve the polynomial equations below

1. $x^4 - 4x^3 + x^2 + 6x = 0$?
2. $x^4 + 14x^3 + 62x^2 = -96x - 72$?
3. $x^3 + 15x^2 + 75x + 125 = 0$?
4. $x^4 + 4x^3 + 17x^2 + 36x + 72 = 0$?
5. $x^4 = x^3 + 22x^2 - 16x - 96$?
6. $x^5 + 6x^4 + 6x^3 = 20x^2 + 39x + 18$?

Press the gray boxes to reveal the roots and their multiplicity.

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